Miniversal deformations of pairs of symmetric forms

Andrii Dmytryshyn

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Abstract

We give a miniversal deformation of each pair of symmetric matrices (A, B) under congruence; that is, a normal form with minimal number of independent parameters to which all matrices $(A+E, B+E')$ close to (A, B) can be reduced by congruence transformations $(A+E, B+E') \mapsto \mathcal{S}(E, E')^T(A+E, B+E')\mathcal{S}(E, E'), \mathcal{S}(0,0) = I$, in which $\mathcal{S}(E, E')$ smoothly depends on the entries of E and E'.

1 Introduction

This is a joint work with Vyacheslav Futorny and Vladimir V. Sergeichuk.

V.I. Arnold [1] pointed out that the reduction of a matrix to its Jordan form is an unstable operation: both the Jordan form and the reduction transformations depend discontinuously on the elements of the original matrix. Therefore, if the elements of a matrix are known only approximately, then it is unwise to reduce it to its Jordan form therefore V. I. Arnold obtained a miniversal deformation of Jordan matrix, i.e. a simplest possible normal form, to which not only a given matrix A, but an arbitrary family of matrices close to it can be reduced by means of a similarity transformation smoothly depending on the elements of A in a neighborhood of zero.

We give the analogous form for a pair of symmetric matrices (earlier we gave it for a pair of skew-symmetric matrices [4]). The problem is important for applications, when the matrices arise as a result of measures, i.e. their entries are given with errors.

(Mini)versal deformation were studied by a various authors in a great number of papers (see [5]).

Outline

In Section 2 we present the main result in terms of holomorphic functions, and in terms of miniversal deformations. We use the canonical matrices of a pair of symmetric forms given by Thompson [6].

Section 3 is a proof of the main result. Firstly the method of constructing deformations is presented and after using it we calculate deformations step by step: for the diagonal blocks, for the off diagonal blocks that correspond to the canonical summands of the same type, and for the off diagonal blocks that correspond to the canonical summands of different types.

Note that in the analogous paper for skew-symmetric matrices [4] there is a section devoted to the constructive proof of the versality of deformations. This section is missed here but it can be done exactly in the same way as in [4].

2 The main theorem

In this section we formulate a theorem about miniversal deformations of pairs of symmetric matrices under congruence (it will be proved in the next section), but first we recall a canonical form of pairs of symmetric matrices under congruence.

Define the $n \times n$ matrices

$$\Lambda_n(\lambda) := \begin{bmatrix} 0 & & & \lambda \\ & & \lambda & 1 \\ & \iddots & \iddots & \\ \lambda & 1 & & 0 \end{bmatrix}, \qquad \Delta_n := \begin{bmatrix} 0 & & & 1 \\ & & 1 & \\ & \iddots & & \\ 1 & & & 0 \end{bmatrix},$$

and the $n \times (n+1)$ matrices

$$F_n := \begin{bmatrix} 1 & 0 & & 0 \\ & \ddots & \ddots & \\ 0 & & 1 & 0 \end{bmatrix}, \qquad G_n := \begin{bmatrix} 0 & 1 & & 0 \\ & \ddots & \ddots & \\ 0 & & 0 & 1 \end{bmatrix}.$$

The following lemma was proved in [6].

Lemma 2.1. *Every pair of symmetric complex matrices is congruent to a direct sum, determined uniquely up to permutation of summands, of pairs of the form*

$$H_n(\lambda) := (\Delta_n, \Lambda_n(\lambda)), \quad \lambda \in \mathbb{C}, \tag{1}$$

$$K_n := (\Lambda_n(0), \Delta_n), \tag{2}$$

$$L_n := \left(\begin{bmatrix} 0 & F_n^T \\ F_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n^T \\ G_n & 0 \end{bmatrix} \right). \tag{3}$$

2.1 The main theorem in terms of holomorphic functions

Let (A, B) be a given pair of $n \times n$ symmetric matrices. For all pairs of symmetric matrices $(A + E, B + E')$ that are close to (A, B), we give their normal form $\mathcal{A}(E, E')$ with respect to congruence transformations

$$(A + E, B + E') \mapsto \mathcal{S}(E, E')^T (A + E, B + E') \mathcal{S}(E, E'), \tag{4}$$

in which $\mathcal{S}(E, E')$ is holomorphic at 0 (i.e., its entries are power series in the entries of E and E' that are convergent in a neighborhood of 0) and $\mathcal{S}(0, 0)$ is a nonsingular matrix.

Since $\mathcal{A}(0, 0) = \mathcal{S}(0, 0)^T (A, B) \mathcal{S}(0, 0)$, we can take $\mathcal{A}(0, 0)$ equalling the congruence canonical form $(A, B)_{\text{can}}$ of (A, B). Then

$$\mathcal{A}(E, E') = (A, B)_{\text{can}} + \mathcal{D}(E, E'), \tag{5}$$

where $\mathcal{D}(E, E')$ is a pair of matrices that are holomorphic at 0 and $\mathcal{D}(0, 0) = (0, 0)$. In the next theorem we obtain $\mathcal{D}(E, E')$ with the minimal number of nonzero entries that can be attained by using transformations (4).

We use the following notation:

- 0_{mn} is the $m \times n$ zero matrix;
- 0_{mn*} is the $m \times n$ matrix $\begin{bmatrix} & & & 0 \\ & 0_{m-1,n-1} & & \vdots \\ & & & 0 \\ 0 & \dots & 0 & * \end{bmatrix}$;
- $0^{\leftarrow}_{mn}$ is the $m \times n$ matrix $\begin{bmatrix} * & \\ \vdots & 0_{m,n-1} \\ * & \end{bmatrix}$;

- $0^{\rightarrow}_{mn}$ is, respectively, the $m \times n$ matrix $\begin{bmatrix} & * \\ 0_{m,n-1} & \vdots \\ & * \end{bmatrix}$;
- $0^{\urcorner}_{mn}$ is the $m \times n$ matrix

$$\begin{bmatrix} * & \dots & * \\ & 0_{m-1,n-1} & \vdots \\ & & * \end{bmatrix} \quad \text{or} \begin{bmatrix} * & & \\ \vdots & 0_{m-1,n-1} & \\ * & \dots & * \end{bmatrix};$$

- $0^{\nwarrow\nwarrow\nwarrow}_{nn}$ is the $n \times n$ matrix (here and after unspecified entries are zeros)

$$\left[\begin{array}{ccc|c} * & * & & \\ * & \ddots & \ddots & \\ & \ddots & * & * \\ \hline & & * & \\ & & & \\ & & & \end{array}\right] \quad \text{when } n \text{ is even and}$$

$$\left[\begin{array}{ccc|c|c} * & * & & & \\ * & \ddots & \ddots & & \\ & \ddots & * & * & \\ \hline & & * & * & \\ \hline & & & & \\ & & & & \end{array}\right] \quad \text{when } n \text{ is odd;}$$

- $0^{\nwsearrow\nwsearrow\nwsearrow}_{nn}$ is the $n \times n$ matrix

$$\begin{bmatrix} * & * & & \\ * & * & \ddots & \\ & \ddots & \ddots & * \\ & & * & * \end{bmatrix};$$

- $\mathcal{Q}_{nm}$ with $n < m$ is the $n \times m$ matrix

$$\begin{bmatrix} 0 & \dots & 0 & & & & \\ \vdots & & \vdots & & 0 & & \\ 0 & \dots & 0 & * & \dots & * & 0 \end{bmatrix} \qquad (m - n \text{ stars});$$

when $n \geq m$ than $\mathcal{Q}_{nm} = 0$.

Further, we will usually omit the indices m and n.

Our main result is the following theorem, which we reformulate in a more abstract form in Theorem 2.2.

Theorem 2.1. *Let*

$$(A,B)_{\mathrm{can}} = X_1 \oplus \cdots \oplus X_t \tag{6}$$

be a canonical pair of symmetric complex matrices for congruence, in which $X_1,\dots,X_t$ *are pairs of the form* (1)–(3). *Its simplest miniversal deformation can be taken in the form* $(A,B)_{\mathrm{can}} + \mathcal{D}$ *in which* $\mathcal{D}$ *is a* $(0,*)$ *matrix pair (the stars denote independent parameters, up to symmetry, see Remark 2.1) whose matrices are partitioned into blocks conformally to the decomposition* (6)*:*

$$\mathcal{D} = \left(\begin{bmatrix} \mathcal{D}_{11} & \dots & \mathcal{D}_{1t} \\ \vdots & \ddots & \vdots \\ \mathcal{D}_{t1} & \dots & \mathcal{D}_{tt} \end{bmatrix}, \begin{bmatrix} \mathcal{D}'_{11} & \dots & \mathcal{D}'_{1t} \\ \vdots & \ddots & \vdots \\ \mathcal{D}'_{t1} & \dots & \mathcal{D}'_{tt} \end{bmatrix}\right) \tag{7}$$

These blocks are defined as follows. Write

$$\mathcal{D}(X_i) := (\mathcal{D}_{ii}, \mathcal{D}'_{ii}) \tag{8}$$

$$\mathcal{D}(X_i, X_j) := ((\mathcal{D}_{ij}, \mathcal{D}'_{ij}), (\mathcal{D}_{ji}, \mathcal{D}'_{ji})) \quad \textit{if } i < j, \tag{9}$$

(Remaind that $((\mathcal{D}_{ij}, \mathcal{D}'_{ij}), (\mathcal{D}_{ji}, \mathcal{D}'_{ji})) = ((\mathcal{D}_{ij}, \mathcal{D}'_{ij}), (\mathcal{D}^T_{ij}, \mathcal{D}'^T_{ij}))$*, hence we drop the second pair from the notation.)*

then

(i) *The diagonal blocks of* $\mathcal{D}$ *are defined by*

$$\mathcal{D}(H_n(\lambda)) = \left(0, 0^{\nwarrow\!\nwarrow\!\nwarrow}\right) \tag{10}$$

$$\mathcal{D}(K_n) = \left(0^{\nwarrow\!\nwarrow\!\nwarrow}, 0\right) \tag{11}$$

$$\mathcal{D}(L_n) = \left(\begin{bmatrix} 0_* & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0^{\nwarrow\!\!\!\searrow\nwarrow\!\!\!\searrow\nwarrow\!\!\!\searrow} & 0 \\ 0 & 0 \end{bmatrix}\right). \tag{12}$$

(ii) *The off-diagonal blocks of* $\mathcal{D}$ *whose horizontal and vertical strips contain summands of* $(A,B)_{\mathrm{can}}$ *of the same type are defined by*

$$\mathcal{D}(H_n(\lambda),\, H_m(\mu)) = \begin{cases} (0,\ 0) & \textit{if } \lambda \neq \mu \\ (0, 0^{\nwarrow}) & \textit{if } \lambda = \mu \end{cases} \tag{13}$$

$$\mathcal{D}(K_n, K_m) = \left(0^{\nwarrow}, 0\right) \tag{14}$$

$$\mathcal{D}(L_n, L_m) = \left(\begin{bmatrix} 0_* & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0^{\urcorner} & \mathcal{Q} \\ \mathcal{Q}^T & 0 \end{bmatrix}\right). \tag{15}$$

(iii) *The off-diagonal blocks of $\mathcal{D}$ whose horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of different types are defined by:*

$$\mathcal{D}(H_n(\lambda), K_m) = (0,0) \tag{16}$$

$$\mathcal{D}(H_n(\lambda), L_m) = (0, 0^{\leftarrow}) \tag{17}$$

$$\mathcal{D}(K_n, L_m) = \left(\begin{bmatrix} 0^{\rightarrow} & 0 \end{bmatrix}, 0\right). \tag{18}$$

Remark 2.1 (About independency of parameters). A matrix pair $\mathcal{D}$ is symmetric. It means that each $\mathcal{D}_{ij}, i<j$ and each of $\mathcal{D}'_{ij}, i<j$ contain independent parameters and just $\frac{(n+1)n}{2}$ parameters of $\mathcal{D}_{ii}$ and $\mathcal{D}'_{ii}$ (both are $n\times n$ matrices) are independent (i.e. all parameters in the upper triangular parts of matrices and on the main diagonals of $\mathcal{D}$ are independent).

The matrix pair $\mathcal{D}$ from Theorem 2.1 will be constructed in Section 3 as follows. The vector space

$$V := \{C^T (A,B)_{\text{can}} + (A,B)_{\text{can}} C \,|\, C \in \mathbb{C}^{n\times n}\}$$

is the tangent space to the congruence class of $(A,B)_{\text{can}}$ at the point $(A,B)_{\text{can}}$ since

$$(I+\varepsilon C)^T (A,B)_{\text{can}} (I+\varepsilon C) = (A,B)_{\text{can}} + \varepsilon (C^T (A,B)_{\text{can}} + (A,B)_{\text{can}} C)$$
$$+\varepsilon^2 C^T (A,B)_{\text{can}} C$$

for all $n\times n$ matrices C and each $\varepsilon \in \mathbb{C}$. Then $\mathcal{D}$ satisfies the following condition:

$$\mathbb{C}_s^{n\times n} \times \mathbb{C}_s^{n\times n} = V \oplus \mathcal{D}(\mathbb{C}) \tag{19}$$

in which $\mathbb{C}_s^{n\times n}$ is the space of all $n\times n$ symmetric matrices, $\mathcal{D}(\mathbb{C})$ is the vector space of all matrix pairs obtained from $\mathcal{D}$ by replacing its stars by complex numbers. Thus, the number of stars in $\mathcal{D}$ is equal to the codimension of the congruence class of $(A,B)_{\text{can}}$. Lemma 3.2 from the next section ensures that any matrix pair with entries 0 and $*$ that satisfies (19) can be taken as $\mathcal{D}$ in Theorem 2.1.

2.2 The main theorem in terms of miniversal deformations

The notion of a miniversal deformation of a matrix with respect similarity was given by V. I. Arnold [1] (see also [3, § 30B]). This notion is easily extended to matrix pairs with respect to congruence.

A *deformation* of a pair of $n \times n$ matrices (A, B) is a holomorphic mapping $\mathcal{A}$ from a neighborhood $\Lambda \subset \mathbb{C}^k$ of $\vec{0} = (0, \dots, 0)$ to the space of pairs of $n \times n$ matrices such that $\mathcal{A}(\vec{0}) = A$.

Let $\mathcal{A}$ and $\mathcal{B}$ be two deformations of (A, B) with the same parameter space $\mathbb{C}^k$. Then $\mathcal{A}$ and $\mathcal{B}$ are considered as *equal* if they coincide on some neighborhood of $\vec{0}$ (this means that each deformation is a germ); $\mathcal{A}$ and $\mathcal{B}$ are called *equivalent* if the identity matrix I_n possesses a deformation $\mathcal{I}$ such that

$$\mathcal{B}(\vec{\lambda}) = \mathcal{I}(\vec{\lambda})^T \mathcal{A}(\vec{\lambda}) \mathcal{I}(\vec{\lambda}) \tag{20}$$

for all $\vec{\lambda} = (\lambda_1, \dots, \lambda_k)$ in some neighborhood of $\vec{0}$.

Definition 2.1. A deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of a matrix pair (A, B) is called *versal* if every deformation $\mathcal{B}(\mu_1, \dots, \mu_l)$ of (A, B) is equivalent to a deformation of the form $\mathcal{A}(\varphi_1(\vec{\mu}), \dots, \varphi_k(\vec{\mu}))$, where all $\varphi_i(\vec{\mu})$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_i(\vec{0}) = 0$. A versal deformation $\mathcal{A}(\lambda_1, \dots, \lambda_k)$ of (A, B) is called *miniversal* if there is no versal deformation having less than k parameters.

By a $(0, *)$ *matrix pair* we mean a pair $\mathcal{D}$ of matrices whose entries are 0 and $*$. We say that a matrix pair *is of the form* $\mathcal{D}$ if it can be obtained from $\mathcal{D}$ by replacing the stars with complex numbers. Denote by $\mathcal{D}(\mathbb{C})$ the space of all matrix pairs of the form $\mathcal{D}$, and by $\mathcal{D}(\vec{\varepsilon})$ the pair of parametric matrices obtained from $\mathcal{D}$ by replacing each (i, j) star with the parameter ε_{ij}. This means that

$$\mathcal{D}(\mathbb{C}) := \Big(\bigoplus_{(i,j) \in \mathcal{I}_1(\mathcal{D})} \mathbb{C} E_{ij} \Big) \times \Big(\bigoplus_{(i,j) \in \mathcal{I}_2(\mathcal{D})} \mathbb{C} E_{ij} \Big), \tag{21}$$

$$\mathcal{D}(\vec{\varepsilon}) := \Big(\sum_{(i,j) \in \mathcal{I}_1(\mathcal{D})} \varepsilon_{ij} E_{ij}, \sum_{(i,j) \in \mathcal{I}_2(\mathcal{D})} f_{ij} E_{ij} \Big), \tag{22}$$

where

$$\mathcal{I}_1(\mathcal{D}), \mathcal{I}_2(\mathcal{D}) \subseteq \{1, \dots, n\} \times \{1, \dots, n\} \tag{23}$$

are the sets of indices of the stars in the first and second matrices, respectively, of the pair $\mathcal{D}$, and E_{ij} is the elementary matrix whose (i,j)th entry is 1 and the others are 0.

We say that a miniversal deformation of (A,B) is *simplest* if it has the form $(A,B)+\mathcal{D}(\vec{\varepsilon})$, where $\mathcal{D}$ is a $(0,*)$ matrix pair. If $\mathcal{D}$ has no zero entries, then it defines the deformation

$$\mathcal{U}(\vec{\varepsilon}) := \Big(A + \sum_{i,j=1}^{n} \varepsilon_{ij} E_{ij},\ B + \sum_{i,j=1}^{n} \varepsilon_{ij} E_{ij}\Big). \tag{24}$$

Since each matrix pair is congruent to its canonical matrix pair, it suffices to construct miniversal deformations of canonical matrix pairs (a direct sum of the summands (1)-(3)). These deformations are given in the following theorem, which is a stronger form of Theorem 2.1.

Theorem 2.2. *A simplest miniversal deformation of the canonical matrix pair* $(A,B)_{\text{can}}$ *of symmetric matrices for congruence can be taken in the form* $(A,B)_{\text{can}} + \mathcal{D}(\vec{\varepsilon})$, *where* $\mathcal{D}$ *is the* $(0,*)$ *matrix partitioned into blocks* $\mathcal{D}_{ij}$ *as in* (7) *that are defined by* (10) - (18) *in the notation* (8) - (9).

3 Proof of the main theorem

3.1 A method of construction of miniversal deformations

Now we give a method of construction of simplest miniversal deformations, which will be used in the proof of Theorem 2.2.

The deformation (24) is universal in the sense that every deformation $\mathcal{B}(\mu_1,\dots,\mu_l)$ of (A,B) has the form $\mathcal{U}(\vec{\varphi}(\mu_1,\dots,\mu_l))$, where $\varphi_{ij}(\mu_1,\dots,\mu_l)$ are convergent in a neighborhood of $\vec{0}$ power series such that $\varphi_{ij}(\vec{0}) = 0$. Hence every deformation $\mathcal{B}(\mu_1,\dots,\mu_l)$ in Definition 2.1 can be replaced by $\mathcal{U}(\vec{\varepsilon})$, which proves the following lemma.

Lemma 3.1. *The following two conditions are equivalent for any deformation* $\mathcal{A}(\lambda_1,\dots,\lambda_k)$ *of pair of matrices* (A,B)*:*

(i) *The deformation* $\mathcal{A}(\lambda_1,\dots,\lambda_k)$ *is versal.*

(ii) *The deformation* (24) *is equivalent to* $\mathcal{A}(\varphi_1(\vec{\varepsilon}),\dots,\varphi_k(\vec{\varepsilon}))$ *in which all* $\varphi_i(\vec{\varepsilon})$ *are convergent in a neighborhood of* $\vec{0}$ *power series such that* $\varphi_i(\vec{0}) = 0$.

For a pair of n-by-n symmetric matrices (A, B) and C, we define

$$T(A,B) := \{C^T(A,B) + (A,B)C \mid C \in \mathbb{C}^{n\times n}\}. \tag{25}$$

If U is a subspace of a vector space V, then each set $v+U$ with $v \in V$ is called a *coset of U in V*.

Lemma 3.2. *Let $(A,B) \in (\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n})$ and let $\mathcal{D}$ be a pair of $(0,*)$-matrices of size $n\times n$. The following are equivalent:*

(i) *The deformation $(A,B) + \mathcal{D}(E,E')$ defined in (21) is miniversal.*

(ii) *The vector space $(\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n})$ decomposes into the direct sum*

$$(\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n}) = T(A,B) \oplus \mathcal{D}(\mathbb{C}). \tag{26}$$

(iii) *Each coset of $T(A,B)$ in $(\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n})$ contains exactly one matrix of the form $\mathcal{D}$.*

Proof. Define the action of the group $GL_n(\mathbb{C})$ of nonsingular n-by-n matrices on the space $[\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n}]$ by

$$(A,B)^S = S^T(A,B)S, \qquad (A,B) \in [\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n}], \quad S \in GL_n(\mathbb{C}).$$

The orbit $(A,B)^{GL_n}$ of (A,B) under this action consists of all pairs of symmetric matrices that are congruent to the pair (A,B).

The space $T(A,B)$ is the tangent space to the orbit $(A,B)^{GL_n}$ at the point (A,B) since

$$\begin{aligned}(A,B)^{I+\varepsilon C} &= (I+\varepsilon C)^T(A,B)(I+\varepsilon C)\\ &= (A,B) + \varepsilon(C^T(A,B) + (A,B)C) + \varepsilon^2 C^T(A,B)C\end{aligned}$$

for all n-by-n matrices C and $\varepsilon \in \mathbb{C}$. Hence $\mathcal{D}(\vec{\varepsilon})$ is transversal to the orbit $(A,B)^{GL_n}$ at the point (A,B) if

$$(\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n}) = T(A,B) + \mathcal{D}(\mathbb{C})$$

(see definitions in [3, § 29E]; two subspaces of a vector space are called *transversal* if their sum is equal to the whole space).

This proves the equivalence of (i) and (ii) since a transversal (of the minimal dimension) to the orbit is a (mini)versal deformation [2, Section 1.6]. The equivalence of (ii) and (iii) is obvious. □

Recall that versality of each deformation $(A,B)+\mathcal{D}(\vec{\varepsilon})$ in which $\mathcal{D}$ satisfies (26) means that there exist a deformation $\mathcal{I}(\vec{\varepsilon})$ of the identity matrix such that $\mathcal{D}(\vec{\varepsilon}) = \mathcal{I}(\vec{\varepsilon})^T\mathcal{U}(\vec{\varepsilon})\mathcal{I}(\vec{\varepsilon})$, where $\mathcal{U}(\vec{\varepsilon})$ is defined in (24).

Thus, a simplest miniversal deformation of $(A,B) \in (\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n})$ can be constructed as follows. Let $(T_1,\dots,T_r)$ be a basis of the space $T(A,B)$, and let $(E_1,\dots,E_{\frac{n(n+1)}{2}}; F_1,\dots,F_{\frac{n(n+1)}{2}})$ be the basis of $(\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n})$ consisting of all elementary matrices (E_{ij},F_{ij}). Removing from the sequence $(T_1,\dots,T_r,E_1,\dots,E_{\frac{n(n+1)}{2}},F_1,\dots,F_{\frac{n(n+1)}{2}})$ every pair of matrices that is a linear combination of the preceding matrices, we obtain a new basis $(T_1,\dots,T_r,E_{i_1},\dots,E_{i_k},F_{i_1},\dots,F_{i_m})$ of the space $(\mathbb{C}_s^{n\times n}, \mathbb{C}_s^{n\times n})$. By Lemma 3.2, the deformation

$$\mathcal{A}(\varepsilon_1,\dots,\varepsilon_k,f_1,\dots,f_m) = (A+\varepsilon_1 E_{i_1}+\dots+\varepsilon_k E_{i_k}, B+f_1F_{i_1}+\dots+f_mF_{i_m})$$

is miniversal.

For each pair of $m\times m$ symmetric matrices (M,N) and each pair $n\times n$ symmetric matrices (L,P), define the vector spaces

$$V(M,N) := \{S^T(M,N)+(M,N)S \mid S\in\mathbb{C}^{m\times m}\} \tag{27}$$

$$\begin{aligned} V((M,N),(L,P)) := \{(R^T(L,P)+(M,N)S, S^T(M,N)+(L,P)R) \mid \\ S\in\mathbb{C}^{m\times n}, R\in\mathbb{C}^{n\times m}\} \end{aligned} \tag{28}$$

Lemma 3.3. *Let $(A,B)=(A_1,B_1)\oplus\dots\oplus(A_t,B_t)$ be a block-diagonal matrix in which every (A_i,B_i) is $n_i\times n_i$. Let $\mathcal{D}$ be a pair of $(0,*)$-matrices having the size of (A,B). Partition it into blocks $(\mathcal{D}_{ij},\mathcal{D}'_{ij})$ conformably to the partition of (A,B) (see (7)). Then $(A,B)+\mathcal{D}(E,E')$ is a simplest miniversal deformation of (A,B) for congruence if and only if*

(i) *every coset of $V(A_i,B_i)$ in $(\mathbb{C}_s^{n_i\times n_i}, \mathbb{C}_s^{n_i\times n_i})$ contains exactly one matrix of the form $(\mathcal{D}_{ii},\mathcal{D}'_{ii})$, and*

(ii) *every coset of $V((A_i,B_i),(A_i^T,B_i^T))$ in $(\mathbb{C}^{n_i\times n_j},\mathbb{C}^{n_i\times n_j})\oplus(\mathbb{C}^{n_j\times n_i},\mathbb{C}^{n_j\times n_i})$ contains exactly two pair of matrices $((Q_1,Q_2),(W_1,W_2))$ in which (Q_1,Q_2) is of the form $(\mathcal{D}_{ij},\mathcal{D}'_{ij})$, (W_1,W_2) is of the form $(\mathcal{D}_{ji},\mathcal{D}'_{ji}) = (\mathcal{D}_{ij}^T,\mathcal{D}'^T_{ij})$.*

Proof. By Lemma 3.2(iii), $(A,B)+\mathcal{D}(\vec{\varepsilon})$ is a simplest miniversal deformation of (A,B) if and only if for each $(C,C')\in(\mathbb{C}_s^{n\times n},\mathbb{C}_s^{n\times n})$ the coset $(C,C')+T(A,B)$ contains exactly one (D,D') of the form $\mathcal{D}$; that is, exactly one

$$(D,D')=(C,C')+S^T(A,B)+(A,B)S\in\mathcal{D}(\mathbb{C})\qquad\text{with } S\in\mathbb{C}^{n\times n}. \tag{29}$$

Partition (D,D'), (C,C'), and S into blocks conformably to the partition of (A,B). By (29), for each i we have $(D_{ii},D'_{ii})=(C_{ii},C'_{ii})+S_{ii}^T(A_i,B_i)+(A_i,B_i)S_{ii}$, and for all i and j such that $i<j$ we have

$$\begin{aligned}\left(\begin{bmatrix}D_{ii}&D_{ij}\\D_{ji}&D_{jj}\end{bmatrix},\begin{bmatrix}D'_{ii}&D'_{ij}\\D'_{ji}&D'_{jj}\end{bmatrix}\right)&=\left(\begin{bmatrix}C_{ii}&C_{ij}\\C_{ji}&C_{jj}\end{bmatrix},\begin{bmatrix}C'_{ii}&C'_{ij}\\C'_{ji}&C'_{jj}\end{bmatrix}\right)\\&+\begin{bmatrix}S_{ii}^T&S_{ji}^T\\S_{ij}^T&S_{jj}^T\end{bmatrix}\left(\begin{bmatrix}A_i&0\\0&A_j\end{bmatrix},\begin{bmatrix}B_i&0\\0&B_j\end{bmatrix}\right)+\left(\begin{bmatrix}A_i&0\\0&A_j\end{bmatrix},\begin{bmatrix}B_i&0\\0&B_j\end{bmatrix}\right)\begin{bmatrix}S_{ii}&S_{ij}\\S_{ji}&S_{jj}\end{bmatrix}.\end{aligned} \tag{30}$$

Thus, (29) is equivalent to the conditions

$$(D_{ii},D'_{ii})=(C_{ii},C'_{ii})+S_{ii}^T(A_i,B_i)+(A_i,B_i)S_{ii}\in\mathcal{D}_{ii}(\mathbb{C}),(1\leqslant i\leqslant t) \tag{31}$$

$$\begin{aligned}&((D_{ij},D'_{ij}),(D_{ji},D'_{ji}))=((C_{ij},C'_{ij}),(C_{ji},C'_{ji}))+\\&((S_{ji}^TA_j+A_iS_{ij},S_{ji}^TB_j+B_iS_{ij}),(S_{ij}^TA_i+A_jS_{ji},S_{ij}^TB_i+B_jS_{ji}))\in\mathcal{D}_{ij}(\mathbb{C})\oplus\mathcal{D}_{ji}(\mathbb{C})\\&\qquad(1\leqslant i<j\leqslant t)\end{aligned} \tag{32}$$

Hence for each $(C,C')\in(\mathbb{C}_s^{n\times n},\mathbb{C}_s^{n\times n})$ there exists exactly one $(D,D')\in\mathcal{D}$ of the form (29) if and only if

(i′) for each $(C_{ii},C'_{ii})\in(\mathbb{C}_s^{n_i\times n_i},\mathbb{C}_s^{n_i\times n_i})$ there exists exactly one $(D_{ii},D'_{ii})\in\mathcal{D}_{ii}$ of the form (31), and

(ii′) for each $((C_{ij},C'_{ij}),(C_{ji},C'_{ji}))\in(\mathbb{C}^{n_i\times n_j},\mathbb{C}^{n_i\times n_j})\oplus(\mathbb{C}^{n_j\times n_i},\mathbb{C}^{n_j\times n_i})$ there exists exactly one $((D_{ij},D'_{ij}),(D_{ji},D'_{ji}))\in\mathcal{D}_{ij}(\mathbb{C})\oplus\mathcal{D}_{ji}(\mathbb{C})$ of the form (32).

This proves the lemma. ∎

Corollary 3.1. In the notation of Lemma 3.3, $(A,B)+\mathcal{D}(\vec{\varepsilon})$ is a miniversal deformation of (A,B) if and only if each submatrix of the form

$$\left(\begin{bmatrix}A_i+\mathcal{D}_{ii}(\vec{\varepsilon})&\mathcal{D}_{ij}(\vec{\varepsilon})\\\mathcal{D}_{ji}(\vec{\varepsilon})&A_j+\mathcal{D}_{jj}(\vec{\varepsilon})\end{bmatrix}\begin{bmatrix}B_i+\mathcal{D}'_{ii}(\vec{\varepsilon})&\mathcal{D}'_{ij}(\vec{\varepsilon})\\\mathcal{D}'_{ji}(\vec{\varepsilon})&B_j+\mathcal{D}'_{jj}(\vec{\varepsilon})\end{bmatrix}\right),\qquad i<j$$

is a miniversal deformation of the pair $(A_i\oplus A_j,B_i\oplus B_j)$.

Let us start to prove Theorem 2.1. Each X_i in (6) is of the form $H_n(\lambda)$, or K_n, or L_n, and so there are 9 types of pairs $\mathcal{D}(X_i)$ and $\mathcal{D}(X_i, X_j)$ with $i < j$; they are given (10)–(18). It suffices to prove that the pairs (10)–(18) satisfy the conditions (i) and (ii) of Lemma 3.3.

3.2 Diagonal blocks of matrices of $\mathcal{D}$

Fist we verify that the diagonal blocks of $\mathcal{D}$ defined in part (i) of Theorem 2.1 satisfy the condition (i) of Lemma 3.3.

3.2.1 Diagonal blocks $\mathcal{D}(H_n(\lambda))$ and $\mathcal{D}(K_n)$

At first consider the pair of blocks $H_n(\lambda)$. The deformation of K_n is equal to the deformation of $H_n(\lambda)$ for $\lambda = 0$ up to the permutation of matrices.

Due to Lemma 3.3(i), it suffices to prove that each pair of symmetric n-by-n matrices (A, B) can be reduced to exactly one pair of matrices of the form (10) (or, respectively (11)) by adding

$$\Delta(A,B) = C^T(\Delta_n, \Lambda_n(\lambda)) + (\Delta_n, \Lambda_n(\lambda))C = (C^T\Delta_n + \Delta_n C, C^T\Lambda_n(\lambda) + \Lambda_n(\lambda)C)$$

in which C is an arbitrary n-by-n matrix.

Obviously, that adding $C^T\Delta_n + \Delta_n C$ one can reduce A to zero. To preserve A we must hereafter take C such that $C^T\Delta_n + \Delta_n C = 0$. This means that C is a skew-symmetric matrix with respect to its skew diagonal.

We reduce B by adding

$$
\begin{aligned}
\Delta B &= C^T\Lambda_n(\lambda) + \Lambda_n(\lambda)C \\
&= \begin{bmatrix} s_{11} & s_{12} & \dots & s_{1n-2} & s_{1n-1} & 0 \\ s_{21} & s_{22} & \dots & s_{2n-2} & 0 & -s_{1n-1} \\ s_{31} & s_{32} & \dots & 0 & -s_{2n-2} & -s_{1n-2} \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ s_{n-11} & 0 & \dots & -s_{32} & -s_{22} & -s_{12} \\ 0 & -s_{n-11} & \dots & -s_{31} & -s_{21} & -s_{11} \end{bmatrix} \begin{bmatrix} 0 & & & & & \lambda \\ & & & & \lambda & 1 \\ & & & \lambda & 1 & \\ & & \iddots & \iddots & & \\ & \lambda & 1 & & & \\ \lambda & 1 & & & & 0 \end{bmatrix} \\
&+ \begin{bmatrix} 0 & & & & & \lambda \\ & & & & \lambda & 1 \\ & & & \lambda & 1 & \\ & & \iddots & \iddots & & \\ & \lambda & 1 & & & \\ \lambda & 1 & & & & 0 \end{bmatrix} \begin{bmatrix} s_{11} & s_{21} & \dots & s_{n-21} & s_{n-11} & 0 \\ s_{12} & s_{22} & \dots & s_{n-22} & 0 & -s_{n-11} \\ s_{13} & s_{23} & \dots & 0 & -s_{n-22} & -s_{n-21} \\ \vdots & \vdots & \iddots & \vdots & \vdots & \vdots \\ s_{1n-1} & 0 & \dots & -s_{23} & -s_{22} & -s_{21} \\ 0 & -s_{1n-1} & \dots & -s_{13} & -s_{12} & -s_{11} \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 & s_{n-11} & s_{n-21} & \dots & s_{31} & s_{21} \\ 0 & -2s_{n-11} & -s_{n-21} & s_{n-22}-s_{41} & \dots & s_{32}-s_{21} & s_{22}-s_{11} \\ s_{n-11} & -s_{n-21} & -2s_{n-22} & -s_{42} & \dots & s_{33}-s_{22} & s_{23}-s_{12} \\ s_{n-21} & s_{n-22}-s_{41} & -s_{42} & -2s_{43} & \dots & s_{34}-s_{23} & s_{24}-s_{13} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{31} & s_{32}-s_{21} & s_{33}-s_{22} & s_{34}-s_{23} & \dots & -2s_{2n-2} & -s_{1n-2} \\ s_{21} & s_{22}-s_{11} & s_{23}-s_{12} & s_{24}-s_{13} & \dots & -s_{1n-2} & -2s_{1n-1} \end{bmatrix} \quad (33)
\end{aligned}
$$

Each skew diagonal of ΔB has unique variables thus we reduce each skew diagonal of B independently. Starting from the lower right hand corner for each of $n-1$ skew diagonals we have a system of equations which has a solution by the Kronecker-Capelli theorem but for each half of the fist n skew diagonals we have a system of equations with the matrix

$$
\begin{pmatrix} 1 & & & & x_1 \\ -1 & 1 & & & x_2 \\ & \ddots & \ddots & & \vdots \\ & & -1 & 1 & x_{k-1} \\ & & & -1 & x_k \end{pmatrix} \text{for even skew diagonals,} \qquad (34)
$$

$$
\begin{pmatrix}
1 & & & & x_1 \\
-1 & 1 & & & x_2 \\
 & \ddots & \ddots & & \vdots \\
 & & -1 & 1 & x_{k-1} \\
 & & & -2 & x_k
\end{pmatrix}
\text{for odd skew diagonals,} \tag{35}
$$

where $x_1 \dots x_k$ are corresponding elements of B. The matrix of this system has $k-1$ columns and k rows, where $1 \leq k \leq [\frac{n}{2}]$, when n is even, and $1 \leq k \leq [\frac{n}{2}]+1$, when n is odd. Hence its rank is less or equal to $k-1$. But the rank of the extended matrix of the system is k, by the Kronecker-Capelli theorem this system does not have a solution. If we drop the first or the last equation of the system then it will have a solution. We should drop the last equation because in that case we can set more elements to zero (on even skew diagonals). Our result does not depend on λ therefore $\mathcal{D}(H_n(\lambda)) = (0, 0^{\nwarrow\nwarrow\nwarrow})$ and $\mathcal{D}(K_n) = (0^{\nwarrow\nwarrow\nwarrow}, 0)$.

3.2.2 Diagonal blocks $\mathcal{D}(L_n)$

In the same way (using Lemma 3.3(i)) we prove that each pair $(A, B) = \left(\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, \begin{bmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{bmatrix} \right)$ of symmetric $2n+1$-by-$2n+1$ matrices can be reduced to the (12) by adding

$$
\begin{aligned}
\Delta(A,B) &= \left(\begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}, \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} \right) \\
&= \begin{bmatrix} S_{11}^T & S_{21}^T \\ S_{12}^T & S_{22}^T \end{bmatrix} \left(\begin{bmatrix} 0 & F_n^T \\ F_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n^T \\ G_n & 0 \end{bmatrix} \right) \\
&+ \left(\begin{bmatrix} 0 & F_n^T \\ F_n & 0 \end{bmatrix}, \begin{bmatrix} 0 & G_n^T \\ G_n & 0 \end{bmatrix} \right) \begin{bmatrix} S_{11} & S_{12} \\ S_{21} & S_{22} \end{bmatrix} \\
&= \left(\begin{bmatrix} S_{21}^T F_n + F_n^T S_{21} & S_{11}^T F_n^T + F_n^T S_{22} \\ S_{22}^T F_n + F_n S_{11} & S_{12}^T F_n^T + F_n S_{12} \end{bmatrix}, \right. \\
&\qquad \left. \begin{bmatrix} S_{21}^T G_n + G_n^T S_{21} & S_{11}^T G_n^T + G_n^T S_{22} \\ S_{22}^T G_n + G_n S_{11} & S_{12}^T G_n^T + G_n S_{12} \end{bmatrix} \right)
\end{aligned} \tag{36}
$$

in which $S = [S_{ij}]_{i,j=1}^2$ is an arbitrary $2n+1 \times 2n+1$ matrix.

Each pair of blocks of our pair of matrices is changed independently.The firth one is $(S_{21}^T F_n + F_n^T S_{21}, S_{21}^T G_n + G_n^T S_{21})$ in which S_{21} is an arbitrary n-by-$n+1$ matrix. Obviously, that adding $\Delta A_{11} = S_{21}^T F_n + F_n^T S_{21}$ one can reduce

each $n+1$-by-$n+1$ symmetric matrix A_{11} to 0_*. To preserve A_{11} we must hereafter take S_{21} such that $F_n^T S_{21} = -S_{21}^T F_n$. This means that

$$S_{21} = \begin{bmatrix} 0 & s_{12} & s_{13} & \dots & s_{1n} & 0 \\ -s_{12} & 0 & s_{23} & \dots & s_{2n} & 0 \\ -s_{13} & -s_{23} & 0 & \dots & s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ -s_{1n} & -s_{2n} & -s_{3n} & \dots & 0 & 0 \end{bmatrix}.$$

The matrix S_{21} without the last column is skew-symmetric. Now we reduce B_{11} by adding

$$\Delta B_{11} = \begin{bmatrix} 0 & -s_{12} & -s_{13} & \dots & -s_{1n} \\ s_{12} & 0 & -s_{23} & \dots & -s_{2n} \\ s_{13} & s_{23} & 0 & \dots & -s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & s_{12} & s_{13} & \dots & s_{1n} & 0 \\ -s_{12} & 0 & s_{23} & \dots & s_{2n} & 0 \\ -s_{13} & -s_{23} & 0 & \dots & s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & 0 \\ -s_{1n} & -s_{2n} & -s_{3n} & \dots & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & -s_{12} & \dots & & -s_{1n-1} & -s_{1n} \\ 0 & 2s_{12} & s_{13} & \dots & & s_{1n} - s_{2n-1} & -s_{2n} \\ -s_{12} & s_{13} & 2s_{23} & \ddots & & s_{2n} - s_{3n-1} & -s_{3n} \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots & \vdots \\ -s_{1n-2} & s_{1n-1} - s_{2n-2} & & \ddots & 2s_{n-2n-1} & s_{n-2n} & -s_{n-1n} \\ -s_{1n-1} & s_{1n} - s_{2n-1} & & \dots & s_{n-2n} & 2s_{n-1n} & 0 \\ -s_{1n} & -s_{2n} & & \dots & -s_{n-1n} & 0 & 0 \end{bmatrix}$$

Both upper and lower parts (with respect to its skew diagonal) of matrix ΔB_{11} are analogous to the upper part of (33). So each skew diagonal of ΔB_{11} has unique variables, hence we reduce B_{11} skew diagonal by skew diagonal to the form $0^{\nwarrow\searrow}$.

The pair of block (A_{21}, B_{21}) is reduced by adding $\Delta(A_{21}, B_{21}) = (S_{22}^T F_n + F_n S_{11}, S_{22}^T G_n + G_n S_{11})$ in which S_{11} and S_{22} are arbitrary matrices of the

corresponding size. Obviously, that adding $S_{22}^T F_n + F_n S_{11}$ we reduce A_{21} to zero. To preserve A_{21} we must hereafter take S_{11} and S_{22} such that $F_n S_{11} = -S_{22}^T F_n$. This means,that

$$S_{11} = \begin{bmatrix} & & & 0 \\ & -S_{22}^T & & 0 \\ & & & \vdots \\ & & & 0 \\ -y_1 & -y_2 & \dots & -y_{n+1} \end{bmatrix}.$$

We reduce B_{12} by adding

$$\Delta B_{12} = S_{22}^T G_n + G_n S_{11} = \begin{bmatrix} s_{11} & s_{12} & s_{13} & \dots & s_{1n} \\ s_{21} & s_{22} & s_{23} & \dots & s_{2n} \\ s_{31} & s_{32} & s_{33} & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nn} \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -s_{11} & -s_{12} & -s_{13} & \dots & -s_{1n} & 0 \\ -s_{21} & -s_{22} & -s_{23} & \dots & -s_{2n} & 0 \\ -s_{31} & -s_{32} & -s_{33} & \dots & -s_{3n} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{n1} & -s_{n2} & -s_{n3} & \dots & -s_{nn} & 0 \\ y_1 & y_2 & y_3 & \dots & y_n & y_{n+1} \end{bmatrix}$$

$$= \begin{bmatrix} -s_{21} & s_{11} - s_{22} & s_{12} - s_{23} & \dots & s_{1n-1} - s_{2n} & s_{1n} \\ -s_{31} & s_{21} - s_{32} & s_{22} - s_{33} & \dots & s_{2n-1} - s_{3n} & s_{2n} \\ -s_{41} & s_{31} - s_{42} & s_{32} - s_{43} & \dots & s_{3n-1} - s_{4n} & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{n1} & s_{n-11} - s_{n2} & s_{n-12} - s_{n3} & \dots & s_{nn-1} - s_{nn} & s_{n-1n} \\ y_1 & s_{n1} + y_2 & s_{n2} + y_3 & \dots & s_{nn+1} + y_n & s_{nn} + y_{n+1} \end{bmatrix}. \quad (37)$$

It is easily seen that we can set B_{12} to zero by adding ΔB_{12} (diagonal by diagonal).

The pair of blocks (A_{12}, B_{12}) is equal to the transposition of (A_{21}, B_{21}).

To the pair of blocks (A_{22}, B_{22}) we can add $\Delta(A_{22}, B_{22}) = (S_{12}^T F_n^T + F_n S_{12}, S_{12}^T G_n^T + G_n S_{12})$ in which S_{12} is an arbitrary $n+1$-by-n matrix. Obviously, that adding $S_{21}^T F_n^T + F_n S_{21}$ one can reduce each n-by-n symmetric matrix A_{22} to zero. To preserve A_{22} we must hereafter take S_{21} such that

$F_n S_{12} = -S_{12}^T F_n^T$. This means that

$$S_{12} = \begin{bmatrix} 0 & s_{12} & s_{13} & \dots & s_{1n} \\ -s_{12} & 0 & s_{23} & \dots & s_{2n} \\ -s_{13} & -s_{23} & 0 & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_{1n} & -s_{2n} & -s_{3n} & \dots & 0 \\ s_{1n+1} & s_{2n+1} & s_{3n+1} & \dots & s_{nn+1} \end{bmatrix}.$$

The matrix S_{12} without the last row is skew-symmetric. Now we reduce B_{22} by adding

$$\Delta B_{22} \begin{bmatrix} 0 & -s_{12} & -s_{13} & \dots & -s_{1n} & s_{1n+1} \\ s_{12} & 0 & -s_{23} & \dots & -s_{2n} & s_{2n+1} \\ s_{13} & s_{23} & 0 & \dots & -s_{3n} & s_{3n+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{1n} & s_{2n} & s_{3n} & \dots & 0 & s_{nn+1} \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & \dots & 0 \\ 1 & 0 & 0 & \ddots & \vdots \\ 0 & 1 & 0 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & 0 & 0 & 1 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \ddots & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & 1 & 0 \\ 0 & \dots & 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & s_{12} & s_{13} & \dots & s_{1n} \\ -s_{12} & 0 & s_{23} & \dots & s_{2n} \\ -s_{13} & -s_{23} & 0 & \dots & s_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -s_{1n} & -s_{2n} & -s_{3n} & \dots & 0 \\ s_{1n+1} & s_{2n+1} & s_{3n+1} & \dots & s_{nn+1} \end{bmatrix}$$

$$= \begin{bmatrix} -2s_{12} & -s_{13} & \dots & -s_{1n} + s_{2n-1} & s_{1n+1} + s_{2n} \\ -s_{13} & -2s_{23} & \dots & -s_{2n} + s_{3n-1} & s_{2n+1} + s_{3n} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -s_{1n} + s_{2n-1} & -s_{2n} + s_{3n-1} & \dots & -2s_{n-1n} & s_{n-1n+1} \\ s_{1n+1} + s_{2n} & s_{2n+1} + s_{3n} & \dots & s_{n-1n+1} & 2s_{nn+1} \end{bmatrix}$$

Each skew diagonal of ΔB_{22} has unique variables thus we reduce B_{22} skew diagonal by skew diagonal. For each half of any skew diagonal we have the system of equations, which has a solution, by the Kronecker-Capelli theorem. Therefore we reduce each skew-diagonal to zero and so we reduce (A_{22}, B_{22}) to zero.

Hence $\mathcal{D}(L_m)$ is equal to (12).

3.3 Off-diagonal blocks of matrices of $\mathcal{D}$ that correspond to summands of $(A, B)_{\mathbf{can}}$ of the same type

Now we verify the condition (ii) of Lemma 3.3 for off-diagonal blocks of $\mathcal{D}$ defined in Theorem 2.1(ii); the diagonal blocks of their horizontal and vertical strips contain summands of $(A, B)_{\text{can}}$ of the same type.

3.3.1 Pairs of blocks $\mathcal{D}(H_n(\mu), H_m(\lambda))$ and $\mathcal{D}(K_n, K_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A, B), (A^T, B^T))$ can be reduced to exactly one group of the form (13) (or, respectively (14)) by adding

$$(R^T H_m(\lambda) + H_n(\mu)S, S^T H_n(\mu) + H_m(\lambda)R), \quad S \in \mathbb{C}^{n\times m}, R \in \mathbb{C}^{m\times n}.$$

Obviously, that we can reduce only (A, B) the second pair $(A^T, B^T))$ is reduced automatically.

$$\Delta(A, B) = R^T H_m(\lambda) + H_n(\mu)S = (R^T \Delta_m + \Delta_n S, R^T \Lambda_m(\lambda) + \Lambda_n(\mu)S).$$

It is clear that we can set A to zero. To preserve A we must hereafter take R and S such that

$$R^T \Delta_m + \Delta_n S = 0 \Leftrightarrow R^T = -\Delta_n S \Delta_m.$$

It follows that B is reduced by adding

$$\begin{aligned} \Delta B &= R^T \Lambda_m(\lambda) + \Lambda_n(\mu)S = -\Delta_n S \Delta_m \Lambda_m(\lambda) + \Lambda_n(\mu)S \\ &= \begin{cases} (\lambda - \mu)s_{n-i+1,j} - s_{n-i+1,j-1} - s_{n-i+2,j} & \text{if} \quad 2 \le i \le n, \quad 2 \le j \le m \\ (\lambda - \mu)s_{n-i+1,j} - s_{n-i+1,j-1} & \text{if} \quad 2 \le j \le m, \quad i = 1 \\ (\lambda - \mu)s_{n-i+1,j} + s_{n-i+2,j} & \text{if} \quad 2 \le i \le n, \quad j = 1 \\ (\lambda - \mu)s_{n1} & \text{if} \quad i = j = 1 \end{cases} . \end{aligned} \tag{38}$$

We have the system of nm equations that has a solution if $\lambda \neq \mu$. Hence in the case $\lambda \neq \mu$ we can set any pair of n-by-m matrices to zero.

Further we look at the case $\lambda = \mu$ then

$$\Delta B = R^T \Lambda_m(\lambda) + \Lambda_n(\lambda) S = -\Delta_n S \Delta_m \Lambda_m(\lambda) + \Lambda_n(\lambda) S$$

$$= \begin{bmatrix} 0 & -s_{51} & -s_{52} & -s_{53} & -s_{54} & -s_{55} & -s_{56} \\ s_{51} & s_{52}-s_{41} & s_{53}-s_{42} & s_{54}-s_{43} & s_{55}-s_{44} & s_{56}-s_{45} & s_{57}-s_{46} \\ s_{41} & s_{42}-s_{31} & s_{43}-s_{32} & s_{44}-s_{33} & s_{45}-s_{34} & s_{46}-s_{35} & s_{47}-s_{36} \\ s_{31} & s_{32}-s_{21} & s_{33}-s_{22} & s_{34}-s_{23} & s_{35}-s_{24} & s_{36}-s_{25} & s_{37}-s_{26} \\ s_{21} & s_{22}-s_{11} & s_{23}-s_{12} & s_{24}-s_{13} & s_{25}-s_{14} & s_{26}-s_{15} & s_{27}-s_{16} \end{bmatrix} \quad (39)$$

We reduce B by adding ΔB diagonal by diagonal to the form $0^{\nwarrow}$.

We prove that $\mathcal{D}(H_m(\mu),\, H_n(\lambda))$ is equal to (13) and respectively $\mathcal{D}(K_m,\, K_n)$ is equal to (14).

3.3.2 Pairs of blocks $\mathcal{D}(L_n, L_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A, B), (A^T, B^T))$ can be reduced to exactly one group of the form (15) by adding

$$(R^T L_m + L_n S, S^T L_n + L_m R), \quad S \in \mathbb{C}^{2n+1 \times 2m+1},\ R \in \mathbb{C}^{2m+1 \times 2n+1}.$$

Obviously, that we can reduce only (A, B) and the pair (A^T, B^T) is reduced automatically.

$$\Delta(A, B) = \left(\begin{bmatrix} \Delta A_{11} & \Delta A_{12} \\ \Delta A_{21} & \Delta A_{22} \end{bmatrix}, \begin{bmatrix} \Delta B_{11} & \Delta B_{12} \\ \Delta B_{21} & \Delta B_{22} \end{bmatrix} \right) = R^T L_m + L_n S$$

$$= \left(R^T \begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & F_n^T \\ F_n & 0 \end{bmatrix} S, R^T \begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix} + \begin{bmatrix} 0 & G_n^T \\ G_n & 0 \end{bmatrix} S \right)$$

$$= \left(\begin{bmatrix} R_{12}^T F_m + F_n^T S_{21} & R_{11}^T F_m^T + F_n^T S_{22} \\ R_{22}^T F_m + F_n S_{11} & R_{21}^T F_m^T + F_n S_{12} \end{bmatrix}, \begin{bmatrix} R_{12}^T G_m + G_n^T S_{21} & R_{11}^T G_m^T + G_n^T S_{22} \\ R_{22}^T G_m + G_n S_{11} & R_{21}^T G_m^T + G_n S_{12} \end{bmatrix} \right).$$

Firstly we reduce the pair (A_{11}, B_{11}). Easy to see that by adding ΔA_{11} we can reduce A_{11} to 0_*. To preserve A_{11} we must hereafter take R_{12} and S_{21} such that $R_{12}^T F_m = -F_n^T S_{21}$. This means

$$R_{12}^T = \begin{bmatrix} & -Q & \\ 0 & \dots & 0 \end{bmatrix}, S_{21} = \begin{bmatrix} & 0 \\ Q & \vdots \\ & 0 \end{bmatrix}, \text{where } Q \text{ is any } n\text{-by-}m \text{ matrix.}$$

Hence

$$\Delta B_{11} = R_{12}^T G_m + G_n^T S_{21} = \begin{bmatrix} & -Q & \\ 0 & \dots & 0 \end{bmatrix} G_m + G_n^T \begin{bmatrix} & & 0 \\ Q & & \vdots \\ & & 0 \end{bmatrix} =$$

$$\begin{bmatrix} 0 & -q_{11} & -q_{12} & -q_{13} & \dots & -q_{1m-1} & -q_{1m} \\ q_{11} & q_{12}-q_{21} & q_{13}-q_{22} & q_{14}-q_{23} & \dots & q_{1m}-q_{2m-1} & -q_{2m} \\ q_{21} & q_{22}-q_{31} & q_{23}-q_{32} & q_{24}-q_{33} & \dots & q_{2m}-q_{3m-1} & -q_{3m} \\ q_{31} & q_{32}-q_{41} & q_{33}-q_{42} & q_{34}-q_{43} & \dots & q_{3m}-q_{4m-1} & -q_{4m} \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n-11} & q_{n-12}-q_{n1} & q_{n-13}-q_{n2} & q_{n-14}-q_{n3} & \dots & q_{n-1m}-q_{nm-1} & -q_{nm} \\ q_{n1} & q_{n2} & q_{n3} & q_{n4} & \dots & q_{nm} & 0 \end{bmatrix} \tag{40}$$

By adding ΔB_{11} we can set each element of B_{11} to zero except either the first column and the last row or the first row and the last column.

Now we turn to the second pair of blocks. We can set A_{12} to zero by adding ΔA_{12}. To preserve A_{12} we must hereafter take R_{11} and S_{22} such that $R_{11}^T F_m^T = -F_n^T S_{22}$ thus

$$R_{11}^T = \begin{bmatrix} & & & b_1 \\ & -S_{22} & & \vdots \\ 0 & \dots & 0 & b_{n+1} \end{bmatrix},$$

where S_{22} is any n-by-m matrix. Therefore

$$\Delta B_{12} = R_{11}^T G_m^T + G_n^T S_{22} = \begin{bmatrix} & & & b_1 \\ & -S_{22} & & \vdots \\ 0 & \dots & 0 & b_{n+1} \end{bmatrix} G_m^T + G_n^T S_{22} =$$

$$\begin{bmatrix} -s_{12} & -s_{13} & -s_{14} & \dots & -s_{1m} & b_1 \\ s_{11}-s_{22} & s_{12}-s_{23} & s_{13}-s_{24} & \dots & s_{1m-1}-s_{2m} & b_2+s_{1m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ s_{n-11}-s_{n2} & s_{n-12}-s_{n3} & s_{n-13}-s_{n4} & \dots & s_{n-1m-1}-s_{nm} & b_n+s_{n-1m} \\ s_{n1} & s_{n2} & s_{n3} & \dots & s_{nm-1} & b_{n+1}+s_{nm} \end{bmatrix} \tag{41}$$

If $n \geq m-1$ then we can set B_{12} to zero by adding ΔB_{12}. If $n < m-1$ then we can not set the whole block B_{12} to zero. We reduce each diagonal of B_{12} independently. By adding the first n diagonals of ΔB_{12} starting from the down left hand corner we set corresponding diagonals of B_{12} to zero. We set the next $m-n-1$ diagonals of B_{12} to zero , except the last elements of every

of them. The last $n+1$ diagonals we set to zero completely. Hence we reduce this pair of blocks to the form $(0, \mathcal{Q}_{n+1m})$.

Now we reduce (A_{21}, B_{21}). We can set A_{21} to zero by adding ΔA_{21}. To preserve A_{21} we must hereafter take R_{22} and S_{11} such that $R_{22}^T F_m = -F_n S_{11}$ thus

$$S_{11} = \begin{bmatrix} & & 0 \\ & -R_{22}^T & \vdots \\ & & 0 \\ b_1 & \dots & b_{m+1} \end{bmatrix},$$

where R_{22}^T is any n-by-m matrix. Therefore

$$\Delta B_{21} = R_{22}^T G_m + G_n S_{11} = R_{22}^T G_m + G_n \begin{bmatrix} & & 0 \\ & -R_{22}^T & \vdots \\ & & 0 \\ b_1 & \dots & b_{m+1} \end{bmatrix}$$

$$= \begin{bmatrix} -r_{21} & r_{11} - r_{22} & r_{12} - r_{23} & \dots & r_{1m-1} - r_{2m} & r_{1m} \\ -r_{31} & r_{21} - r_{32} & r_{22} - r_{33} & \dots & r_{2m-1} - r_{3m} & r_{2m} \\ -r_{41} & r_{31} - r_{42} & r_{32} - r_{43} & \dots & r_{3m-1} - r_{4m} & r_{3m} \\ -r_{51} & r_{41} - r_{52} & r_{42} - r_{53} & \dots & r_{4m-1} - r_{5m} & r_{4m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ -r_{n1} & r_{n-11} - r_{n2} & r_{n-12} - r_{n3} & \dots & r_{n-1m-1} - r_{nm} & r_{n-1m} \\ b_1 & r_{n1} + b_2 & r_{n2} + b_3 & \dots & r_{nm-1} + b_m & r_{nm} + b_{m+1} \end{bmatrix} \qquad (42)$$

If $m+1 \geq n$ then we can set B_{21} to zero by adding ΔB_{21}. If $m+1 < n$ then we can not set the whole B_{21} to zero. We reduce like in the previous case. Hence (A_{21}, B_{21}) is reduced by adding $\Delta(A_{21}, B_{21})$ to $(0, \mathcal{Q}_{m+1n}^T)$.

Now let us consider the pair (A_{22}, B_{22}). We can set A_{22} to zero by adding ΔA_{22}. To preserve A_{22} we must hereafter take R_{21} and S_{12} such that $R_{21}^T F_m^T = -F_n S_{12}$ thus

$$S_{12} = \begin{bmatrix} & -Q & \\ a_1 & \dots & a_m \end{bmatrix}, R_{21}^T = \begin{bmatrix} & b_1 \\ Q & \vdots \\ & b_n \end{bmatrix}, \text{where } Q \text{ is any } n\text{-by-}m \text{ matrix.}$$

It follows that

$$\Delta B_{22} = R_{21}^T G_m^T + G_n S_{12} = \begin{bmatrix} Q & \begin{matrix} b_1 \\ \vdots \\ b_n \end{matrix} \end{bmatrix} G_m^T + G_n \begin{bmatrix} & -Q & \\ a_1 & \dots & a_m \end{bmatrix} =$$

$$\begin{bmatrix} q_{12}-q_{21} & q_{13}-q_{22} & q_{14}-q_{23} & \dots & q_{1m}-q_{2m-1} & b_1-q_{2m} \\ q_{22}-q_{31} & q_{23}-q_{32} & q_{24}-q_{33} & \dots & q_{2m}-q_{3m-1} & b_2-q_{3m} \\ q_{32}-q_{41} & q_{33}-q_{42} & q_{34}-q_{43} & \dots & q_{3m}-q_{4m-1} & b_3-q_{4m} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ q_{n-12}-q_{n1} & q_{n-13}-q_{n2} & q_{n-14}-q_{n3} & \dots & q_{n-1m}-q_{nm-1} & b_{m-1}-q_{nm} \\ q_{n2}+a_1 & q_{n3}+a_2 & q_{n4}+a_3 & \dots & q_{nm}-a_{n-1} & a_n+b_m \end{bmatrix} \qquad (43)$$

We can set each skew-diagonal of B_{22} to zero independently. Thus adding ΔB_{22} we can reduce B_{22} to zero.

Hence $\mathcal{D}(L_m, L_n)$ has the form (15).

3.4 Off-diagonal blocks of matrices of $\mathcal{D}$ that correspond to summands of $(A,B)_{\text{can}}$ of distinct types

Finally, we verify the condition (ii) of Lemma 3.3 for off-diagonal blocks of $\mathcal{D}$ defined in Theorem 2.1(iii); the diagonal blocks of their horizontal and vertical strips contain summands of $(A,B)_{\text{can}}$ of different types.

3.4.1 Pairs of blocks $\mathcal{D}(H_n(\lambda), K_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(A^T,B^T))$ can be reduced to exactly one group of the form (16) by adding

$$(R^T K_m + H_n(\lambda)S, S^T H_n(\lambda) + K_m R), \quad S \in \mathbb{C}^{n\times m},\ R \in \mathbb{C}^{m\times n}.$$

Obviously, that we can reduce only (A,B) and the pair (A^T,B^T) is reduced automatically.

$$\Delta(A,B) = R^T K_m + H_n(\lambda)S = (R^T\Lambda_m(0) + \Delta_n S, R^T\Delta_m + \Lambda_n(\lambda)S).$$

It is clear that we can set A to zero by adding ΔA. To preserve A we must hereafter take R and S such that

$$R^T\Lambda_m(0) + \Delta_n S = 0 \Rightarrow S = -\Delta_n R^T \Lambda_m(0).$$

Thus B is reduced by adding:

$$\Delta B = R^T\Delta_m + \Lambda_n(\lambda)S = R^T\Delta_m - \Lambda_n(\lambda)\Delta_n R^T\Lambda_m(0)$$

We can set B_{22} to zero by adding ΔB_{22}. Hence $\mathcal{D}(H_n(\lambda), K_m)$ is equal to zero.

3.4.2 Pairs of blocks $\mathcal{D}(H_n(\lambda), L_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(A^T,B^T))$ can be reduced to (17) by adding

$$(R^T L_m + H_n(\lambda)S, S^T H_n(\lambda) + L_m R), \quad S \in \mathbb{C}^{n\times 2m+1},\ R \in \mathbb{C}^{2m+1\times n}.$$

Obviously, that we can reduce only (A,B) and the pair (A^T,B^T) is reduced automatically.

$$\Delta(A,B) = R^T L_m + H_n(\lambda)S = (R^T\begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix} + \Delta_n S, R^T\begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix} + \Lambda_n(\lambda)S).$$

It is easy to check that we can set A to zero. To preserve A we must hereafter take R and S such that

$$R^T\begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix} + \Delta_n S = 0 \Rightarrow S = -\Delta_n \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix}\begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix}$$

Hence B is reduced by adding:

$$\Delta B = R^T\begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix} - \Lambda_n(\lambda)\Delta_n R^T\begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix}$$
$$= \begin{cases} -\lambda r_{in-1} - r_{i-1n-1} & \text{if} \quad 1 \le j \le n, \quad j = 1 \\ -\lambda r_{i,m+1+j} - r_{i-1,m+1+j} + r_{i,m+j} & \text{if} \quad 1 \le i \le n, \quad 1 < j < m+1 \\ r_{in} & \text{if} \quad 1 \le i \le n, \quad j = m+1 \\ -\lambda r_{i,j-m-1} - r_{i-1,j-m-1} + r_{i,j-m} & \text{if} \quad 1 \le i \le n, \quad m+1 < j \le 2m+1 \end{cases},$$

where we put $r_{0t} := 0$. Adding ΔB we reduce B to the form $0^{\leftarrow}$. Therefore $\mathcal{D}(H_n(\lambda), L_m)$ is equal to (17).

3.4.3 Pairs of blocks $\mathcal{D}(K_n, L_m)$

Due to Lemma 3.3(ii), it suffices to prove that each group of four matrices $((A,B),(A^T,B^T))$ can be reduced to (18) by adding

$$(R^T L_m + K_n S, S^T K_n + L_m R), \quad S \in \mathbb{C}^{n\times 2m+1},\ R \in \mathbb{C}^{2m+1\times n}.$$

Obviously, that we can reduce only (A,B) and the pair (A^T,B^T) is reduced automatically.

$$\Delta(A,B) = R^T L_m + K_n S = (R^T \begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix} + \Lambda_n(0)S, R^T \begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix} + \Delta_n S).$$

It is easy to check that we can set B to zero. To preserve B we must hereafter take R and S such that

$$R^T \begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix} + \Delta_n S = 0 \Rightarrow S = -\Delta_n \begin{bmatrix} R_{11}^T & R_{21}^T \\ R_{12}^T & R_{22}^T \end{bmatrix} \begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix}$$

Thus A is reduced by adding:

$$\begin{aligned}\Delta A &= R^T \begin{bmatrix} 0 & F_m^T \\ F_m & 0 \end{bmatrix} - \Lambda_n(0)\Delta_n R^T \begin{bmatrix} 0 & G_m^T \\ G_m & 0 \end{bmatrix} \\ &= \begin{cases} r_{in-1} & \text{if} \quad 1 \le j \le n, \quad j = 1 \\ r_{i,m+1+j} - r_{i-1,m+j} & \text{if} \quad 1 \le i \le n, \quad 1 < j < m+1 \\ r_{i-1,n} & \text{if} \quad 1 \le i \le n, \quad j = m+1 \\ r_{i,j-m-1} - r_{i-1,j-m} & \text{if} \quad 1 \le i \le n, \quad m+1 < j \le 2m+1 \end{cases},\end{aligned}$$

where we put $r_{0t} := 0$.

Therefore $\mathcal{D}(K_n, L_m)$ is equal to (18).

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